Time Series
Mining and Forecasting

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Partly based on materials by
Professors Guy Lebanon, Jeffrey Heer, John Stasko, Christos Faloutsos, Parishit Ram (GT PhD alum; SkyTree), Alex Gray
Outline

• Motivation
• Similarity search – distance functions
• Linear Forecasting
• Non-linear forecasting
• Conclusions
Problem definition

• **Given**: one or more sequences
  
  \( x_1, x_2, \ldots, x_t, \ldots \)
  
  \((y_1, y_2, \ldots, y_t, \ldots)\)
  
  \((\ldots)\)

• **Find**
  
  – similar sequences; forecasts
  
  – patterns; clusters; outliers
Motivation - Applications

• Financial, sales, economic series

• Medical
  – ECGs +; blood pressure etc monitoring
  – reactions to new drugs
  – elderly care
Motivation - Applications (cont’d)

• ‘Smart house’
  – sensors monitor temperature, humidity, air quality

• video surveillance
Motivation - Applications (cont’d)

- Weather, environment/anti-pollution
  - volcano monitoring
  - air/water pollutant monitoring

Sunspot Data
Motivation - Applications (cont’d)

• Computer systems
  – ‘Active Disks’ (buffering, prefetching)
  – web servers (ditto)
  – network traffic monitoring
  – ...

Stream Data: Disk accesses

#bytes

time
Problem #1:

Goal: given a signal (e.g., #packets over time)
Find: patterns, periodicities, and/or compress

 lynx caught per year
(packets per day; temperature per day)
Problem#2: Forecast

Given $x_t, x_{t-1}, \ldots$, forecast $x_{t+1}$
Problem #2': Similarity search

E.g., Find a 3-tick pattern, similar to the last one
Problem #3:

- Given: A set of correlated time sequences
- Forecast ‘Sent(t)’
Important observations

Patterns, rules, forecasting and similarity indexing are closely related:

• To do forecasting, we need
  – to find patterns/rules
  – to find similar settings in the past

• to find outliers, we need to have forecasts
  – (outlier = too far away from our forecast)
Outline

• Motivation
• Similarity search and distance functions
  – Euclidean
  – Time-warping
• ...
Importance of distance functions

Subtle, but absolutely necessary:

• A ‘must’ for similarity indexing (-> forecasting)

• A ‘must’ for clustering

Two major families

– Euclidean and Lp norms

– Time warping and variations
Euclidean and Lp

\[ D(\vec{x}, \vec{y}) = \sum_{i=1}^{n} (x_i - y_i)^2 \]

\[ L_p(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|^p \]

\text{L}_1: \text{city-block} = \text{Manhattan} \\
\text{L}_2 = \text{Euclidean} \\
\text{L}_\infty
Observation #1

Time sequence -> n-d vector

Day-1

Day-2

Day-n
Observation #2

Euclidean distance is closely related to
– cosine similarity
– dot product
Time Warping

• allow accelerations - decelerations
  – (with or without penalty)

• THEN compute the (Euclidean) distance (+ penalty)

• related to the string-editing distance
Time Warping

‘stutters’:
Time warping

Q: how to compute it?
A: dynamic programming

\[ D(i, j) = \text{cost to match} \]

prefix of length \( i \) of first sequence \( x \) with prefix of length \( j \) of second sequence \( y \)
Time warping

Time warping

Thus, with no penalty for stutter, for sequences

\[ x_1, x_2, \ldots, x_i,; \quad y_1, y_2, \ldots, y_j \]

\[
D(i, j) = \|x[i] - y[j]\| + \min \left\{ D(i - 1, j - 1), D(i, j - 1), D(i - 1, j) \right\}
\]

- no stutter
- x-stutter
- y-stutter
Time warping

VERY SIMILAR to the string-editing distance

\[
D(i, j) = \|x[i] - y[j]\| + \min \begin{cases} D(i - 1, j - 1) & \text{no stutter} \\ D(i, j - 1) & \text{x-stutter} \\ D(i - 1, j) & \text{y-stutter} \end{cases}
\]
Time warping

- Complexity: $O(M \times N)$ - quadratic on the length of the strings
- Many variations (penalty for stutters; limit on the number/percentage of stutters; ...)
- popular in voice processing
  [Rabiner + Juang]
Other Distance functions

• piece-wise linear/flat approx.; compare pieces [Keogh+01] [Faloutsos+97]
• ‘cepstrum’ (for voice [Rabiner+Juang])
  – do DFT; take log of amplitude; do DFT again!
• Allow for small gaps [Agrawal+95]

See tutorial by [Gunopulos + Das, SIGMOD01]
Other Distance functions

• In [Keogh+, KDD’04]: parameter-free, MDL based
Conclusions

Prevailing distances:

– Euclidean and
– time-warping
Outline

• Motivation
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Linear Forecasting
Outline

• Motivation
• ...
• Linear Forecasting
  – Auto-regression: Least Squares; RLS
  – Co-evolving time sequences
  – Examples
  – Conclusions
Problem #2: Forecast

- Example: give $x_{t-1}$, $x_{t-2}$, ..., forecast $x_t$
Forecasting: Preprocessing

MANUALLY:
- remove trends
- spot periodicities

7 days
Problem#2: Forecast

- Solution: try to express $x_t$ as a linear function of the past: $x_{t-1}, x_{t-2}, \ldots,$ (up to a window of $w$)

Formally:

$$x_t \approx a_1 x_{t-1} + \ldots + a_w x_{t-w} + \text{noise}$$
(Problem: Back-cast; interpolate)

- Solution - interpolate: try to express $x_t$ as a linear function of the past AND the future:

  $x_{t+1}, x_{t+2}, \ldots x_{t+w_{\text{future}}}; x_{t-1}, \ldots x_{t-w_{\text{past}}}$

  (up to windows of $w_{\text{past}}, w_{\text{future}}$)

- EXACTLY the same algo’s
Refresher: Linear Regression

<table>
<thead>
<tr>
<th>patient</th>
<th>weight</th>
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<tbody>
<tr>
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N 25 ??

Express what we don’t know (= “dependent variable”) as a linear function of what we know (= “independent variable(s)”)
Refresher: Linear Regression

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**Lag \( w = 1 \)**

Dependent variable = \# of packets sent (\( S[t] \))

Independent variable = \# of packets sent (\( S[t-1] \))
Linear **Auto** Regression

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'lag-plot'

#packets sent at time \( t \)

#packets sent at time \( t-1 \)

**Lag \( w = 1 \)**

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### Linear Auto Regression

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Lag $w = 1$

- **Dependent variable** = # of packets sent ($S[t]$)
- **Independent variable** = # of packets sent ($S[t-1]$)

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**‘lag-plot’**

#packets sent at time $t$

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Linear Auto Regression

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Lag $w = 1$

Dependent variable = # of packets sent ($S \ [t]$)

Independent variable = # of packets sent ($S[t-1]$)

#packets sent at time $t$

#packets sent at time $t-1$
More details:

- Q1: Can it work with window $w > 1$?
- A1: YES!
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More details:

• Q1: Can it work with window $w > 1$?
• A1: YES! The problem becomes:

$$X_{[N \times w]} \times a_{[w \times 1]} = y_{[N \times 1]}$$

• OVER-CONSTRAINED
  – $a$ is the vector of the regression coefficients
  – $X$ has the $N$ values of the $w$ indep. variables
  – $y$ has the $N$ values of the dependent variable
More details:

- \( X_{[N \times w]} \times a_{[w \times 1]} = y_{[N \times 1]} \)

\[ \begin{align*}
\text{Ind-var1} & \quad \quad \text{Ind-var-w} \\
\downarrow & \quad \quad \uparrow \\
\begin{bmatrix}
X_{11}, X_{12}, \ldots, X_{1w} \\
\hline
X_{21}, X_{22}, \ldots, X_{2w} \\
\vdots \\
\vdots \\
\vdots \\
X_{N1}, X_{N2}, \ldots, X_{Nw}
\end{bmatrix} & \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_w
\end{bmatrix} = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N
\end{bmatrix}
\end{align*} \]
More details:

- \( X_{[N \times w]} \times a_{[w \times 1]} = y_{[N \times 1]} \)

\( \text{Ind-var1} \quad \text{Ind-var-w} \)

\[
\begin{bmatrix}
X_{11}, X_{12}, \ldots, X_{1w} \\
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\times
\begin{bmatrix}
a_1 \\
a_2 \\
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a_w
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N
\end{bmatrix}
\]
More details

• Q2: How to estimate $a_1, a_2, \ldots a_w = a$?
• A2: with Least Squares fit

\[
a = (X^T \times X)^{-1} \times (X^T \times y)
\]

• (Moore-Penrose pseudo-inverse)
• $a$ is the vector that minimizes the RMSE from $y$
More details

• Straightforward solution:

\[ a = \left( X^T \times X \right)^{-1} \times (X^T \times y) \]

- \( a \): Regression Coeff. Vector
- \( X \): Sample Matrix

• Observations:
  - Sample matrix \( X \) grows over time
  - needs matrix inversion
  - \( O(N \times w^2) \) computation
  - \( O(N \times w) \) storage
Even more details

• Q3: Can we estimate a incrementally?
• A3: Yes, with the brilliant, classic method of “Recursive Least Squares” (RLS) (see, e.g., [Yi+00], for details).
• We can do the matrix inversion, WITHOUT inversion! (How is that possible?!)
Even more details

- Q3: Can we estimate a incrementally?
- A3: Yes, with the brilliant, classic method of “Recursive Least Squares” (RLS) (see, e.g., [Yi+00], for details).
- We can do the matrix inversion, WITHOUT inversion! (How is that possible?!) 
- A: our matrix has special form: \((X^T X)\)
More details

At the $N+1$ time tick:

\[ X_{N+1} \quad X_N \]

\[ W \]

\[ N \]

\[ X_{N+1} \]
More details: key ideas

- Let $G_N = (X_N^T \times X_N)^{-1}$ ("gain matrix")
- $G_{N+1}$ can be computed recursively from $G_N$ without matrix inversion
Comparison:

- **Straightforward Least Squares**
  - Needs huge matrix (growing in size) \( O(N \times w) \)
  - Costly matrix operation \( O(N \times w^2) \)

- **Recursive LS**
  - Need much smaller, fixed size matrix \( O(w \times w) \)
  - Fast, incremental computation \( O(1 \times w^2) \)
  - no matrix inversion

\[ N = 10^6, \quad w = 1-100 \]
EVEN more details:

\[
G_{N+1} = G_N - [c]^{-1} \times [G_N \times x_{N+1}^T] \times x_{N+1} \times G_N
\]

Let’s elaborate

(VERY IMPORTANT, VERY VALUABLE!)
EVEN more details:

\[ a = \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \times \left[ X_{N+1}^T \times y_{N+1} \right] \]
EVEN more details:

$$a = \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \times \left[ X_{N+1}^T \times y_{N+1} \right]$$

\[ [w \times 1] \quad [w \times (N+1)] \quad [(N+1) \times w] \quad [w \times (N+1)] \quad [(N+1) \times 1] \]
$a = \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \times \left[ X_{N+1}^T \times y_{N+1} \right]$
EVEN more details:

\[ a = \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \times \left[ X_{N+1}^T \times y_{N+1} \right] \]

\[ G_{N+1} \equiv \left[ X_{N+1}^T \times X_{N+1} \right]^{-1} \]

\[ G_{N+1} = G_N - [c]^{-1} \times \left[ G_N \times x_{N+1}^T \right] \times x_{N+1} \times G_N \]

SCALAR!

\[ c = \left[ 1 + x_{N+1} \times G_N \times x_{N+1}^T \right] \]
Altogether:

\[ G_0 \equiv \delta \cdot I \]

where
I: \(w \times w\) identity matrix
\(\delta\): a large positive number
Comparison:

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  - Needs huge matrix (growing in size)
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$N = 10^6, \quad w = 1-100$
Pictorially:

- Given:

  - Independent Variable
  - Dependent Variable
Pictorially:

- Independent Variable
- Dependent Variable

new point
Pictorially:

RLS: quickly compute new best fit
Even more details

• Q4: can we ‘forget’ the older samples?
• A4: Yes - RLS can easily handle that [Yi+00]:
Adaptability - ‘forgetting’

Dependent Variable
e.g., #bytes sent

Independent Variable
e.g., #packets sent
Adaptability - ‘forgetting’

Independent Variable
eg. #packets sent

Dependent Variable
eg., #bytes sent

Trend change

(R)LS with no forgetting
Adaptability - ‘forgetting’

- RLS: can *trivially* handle ‘forgetting’